

EMBEDDINGS OF NON-POSITIVELY CURVED COMPACT SURFACES IN FLAT LORENTZIAN MANIFOLDS

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ABSTRACT. We prove that any metric of non-positive curvature in the sense of Alexandrov on a compact surface can be isometrically embedded as a convex spacelike Cauchy surface in a flat (2+1) spacetime. The proof follows from polyhedral approximation.

Keywords Alexandrov surfaces, convex surfaces, Minkowski space, Teichmüller space.

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1. INTRODUCTION

In the 1940's, A.D. Alexandrov, looking at the induced (intrinsic) distances on the boundary of convex bodies of the Euclidean space, introduced a class of distances on compact surfaces. Nowadays, such distances are called *metrics of non-negative curvature (in the sense of Alexandrov)*. He then proved the following famous result [Ale06]. We assume that all the surfaces we are considering are closed, oriented and connected.

Theorem 1.1. *Let (S, d) be a metric of non-negative curvature on a compact surface. Then there exists a flat Riemannian manifold R homeomorphic to $S \times \mathbb{R}$ which contains a convex surface whose induced distance is isometric to (S, d) .*

Actually, if (S, d) is a metric space isometric to the induced distance on a flat torus (T, h) , then the statement above is trivial, as R can be taken as $S \times \mathbb{R}$ with the metric $h + dt^2$. Otherwise, by the Gauss–Bonnet formula, a compact surface S with a metric of non-negative curvature must have genus 0. In this case, the metric on R is the one of the Euclidean space minus the origin. A more classical way to state Theorem 1.1 in this case is to say that (S, d) is isometric to the induced distance on the boundary of a convex body of the Euclidean space.

In the present paper, we prove an analogous result for metrics of non-positive curvature.

Theorem 1.2. *Let (S, d) be a metric of non-positive curvature (in the sense of Alexandrov) on a compact surface. Then there exists a flat Lorentzian manifold L homeomorphic to $S \times \mathbb{R}$ which contains a spacelike convex surface whose induced distance is isometric to (S, d) .*

The definition of metric of non-positive curvature is recalled in Section 2.3. Once again, if (S, d) is a metric space isometric to the induced distance on a flat torus (T, h) , then the statement above is trivial, as L can be taken as $S \times \mathbb{R}$ with the metric $h - dt^2$. Otherwise, by Gauss–Bonnet formula, S must have genus $g > 1$. We will now consider only this case.

Actually we will prove Theorem 1.3, which implies Theorem 1.2. Let us give some definitions before stating Theorem 1.3.

The Minkowski space $\mathbb{R}^{2,1}$ is \mathbb{R}^3 endowed with the bilinear form

$$\langle x, y \rangle_- = x_1y_1 + x_2y_2 - x_3y_3 .$$

A plane P is *spacelike* if the restriction of $\langle \cdot, \cdot \rangle_-$ to P is positive definite. A *spacelike convex set* K of $\mathbb{R}^{2,1}$ is a closed convex set which has only spacelike planes as support planes. This assumption is more restrictive than only asking that K is the intersection of half-spaces bounded by spacelike planes, as shows the closure of

$$I^+(0) = \{x \in \mathbb{R}^3 | \langle x, x \rangle_- < 0, x_3 > 0\} .$$

Without loss of generality, we can assume that the set K is *future convex*, i.e. K is the intersection of the future sides of its support planes (the future side of the plane is the one containing the vector $(0, 0, 1)$). A *spacelike convex surface* is the boundary of a (future) convex spacelike set of Minkowski space.

The *induced distance* on a spacelike convex surface is the distance induced by the length structure given by

$$(1) \quad \mathfrak{L}(c) = \int_0^1 \|c'\|_- ,$$

where, for a spacelike vector v ,

$$\|v\|_- = \langle v, v \rangle_-^{1/2},$$

and $c : [0, 1] \rightarrow \partial K$ a Lipschitz curve (with respect to the ambient Euclidean metric of \mathbb{R}^3). Note that as the set is convex, there is always a Lipschitz curve between two points on the boundary.

A famous example is $\{x \in I^+(0) | \langle x, x \rangle_- < -1\}$ whose boundary is

$$(2) \quad \mathbb{H}^2 = \{x \in I^+(0) | \langle x, x \rangle_- = -1\}.$$

If $d_{\mathbb{H}^2}$ is the induced distance on \mathbb{H}^2 , then $(\mathbb{H}^2, d_{\mathbb{H}^2})$ is isometric to the distance of the hyperbolic plane. Moreover, this implies a canonical identification between the group of orientation-preserving isometries of the hyperbolic plane and the connected component of the identity $O_0(1, 2)$ of $O(1, 2)$.

Let us go back to the distance (S, d) of the statement of Theorem 1.2 and to its universal cover (\tilde{S}, \tilde{d}) (see Section 2.2 for the definition of the distance \tilde{d}).

We want to find a *spacelike convex isometric immersion* of (\tilde{S}, \tilde{d}) into $\mathbb{R}^{2,1}$, i.e., a map

$$\phi : \tilde{S} \rightarrow \mathbb{R}^{2,1}$$

such that

- $\phi(\tilde{S})$ is a spacelike convex surface
- ϕ is an isometry between (\tilde{S}, \tilde{d}) and $\phi(\tilde{S})$ endowed with the induced distance.

Moreover we want ϕ to be *equivariant*, that is, there exists a faithful and discrete representation

$$\rho : \pi_1 S \rightarrow \text{Isom } \mathbb{R}^{2,1}$$

such that for all $\gamma \in \pi_1 S$ and for all $x \in \tilde{S}$,

$$(3) \quad \phi(\gamma.x) = \rho(\gamma)\phi(x),$$

where the action of $\pi_1 S$ onto \tilde{S} is by deck transformations.

It follows that the distance induced on $\phi(\tilde{S})/\rho(\pi_1 S)$ is isometric to (S, d) . In the present paper, we will look more precisely for a spacelike convex isometric immersion which is a *Fuchsian convex isometric immersion*. This means that

- $\phi(\tilde{S})$ is contained in $I^+(0)$ (recall that our convex sets are implicitly future convex),
- $\rho(\pi_1 S)$ is a Fuchsian subgroup of $O_0(2, 1)$.

One can easily see that in the Fuchsian case, $\phi(\tilde{S})$ meets exactly once each future timelike half-line.

Let us give a trivial example. If (S, d) is the distance given by a hyperbolic metric h on S , then there is an isometry between (\tilde{S}, \tilde{d}) and \mathbb{H}^2 (the developing map of the hyperbolic structure given by h) and there exists an equivariant representation $\rho : \pi_1 S \rightarrow O_0(2, 1)$ (the holonomy) such that $\mathbb{H}^2/\rho(\pi_1 S)$ with its induced distance is isometric to (S, d) .

In the present paper, we prove the following theorem.

Theorem 1.3. *Let (S, d) be a metric of non-positive curvature on a compact surface (of genus ≥ 2). Then there exists a Fuchsian convex isometric immersion of (\tilde{S}, \tilde{d}) into Minkowski space.*

Theorem 1.3 implies Theorem 1.2: the surface $\phi(\tilde{S})/\rho(\pi_1 S)$ in the flat Lorentzian manifold $L = I^+(0)/\rho(\pi_1 S)$ is isometric to (S, d) . One can moreover precise the statement of Theorem 1.2:

- the flat Lorentzian manifold L contains a totally umbilic hyperbolic surface, namely, $\mathbb{H}^2/\rho(\pi_1 S)$;
- actually, if g is the Riemannian metric of $\mathbb{H}^2/\rho(\pi_1 S)$, the metric of L is $t^2 g - dt^2$;
- the isometric embedding of (S, d) is a *Cauchy surface*, i.e. it meets exactly once each inextensible non spacelike curve of L .

Theorem 1.3 is already known when (S, d) comes from a smooth Riemannian metric on S :

Theorem 1.4 ([LS00]). *Let (S, d) be a smooth Riemannian metric of negative sectional curvature on a compact surface. Then there exists a smooth Fuchsian convex isometric immersion of (\tilde{S}, \tilde{d}) into Minkowski space.*

Moreover, it is proved in [LS00] that the immersion is unique among smooth Fuchsian convex immersions, up to composition by an element of $O(2, 1)$. If the image of the immersion given by Theorem 1.3 is known to be C^1 , then finer results about the regularity of the immersion with respect to the regularity of the metric are available [Sok77]. There also exists a result about isometric immersions into $I^+(0)$ of smooth metrics on the disc of curvature with a negative upper bound [CY15].

Theorem 1.3 is also already known when (S, d) is a *polyhedral* metric of non-positive curvature, which means that (S, d) is a flat metric on S with conical singularities of negative curvature (i.e. the cone angles are $> 2\pi$ at the singular points).

Theorem 1.5 ([Fil11]). *Let (S, d) be a polyhedral metric of non-positive curvature. Then there exists a polyhedral Fuchsian convex isometric immersion of (\tilde{S}, \tilde{d}) into Minkowski space.*

For a variational approach of the above result, see [Bru].

An immersion is *polyhedral* if its image is the boundary of the convex hull (in \mathbb{R}^3) of the orbit for $\rho(\pi_1 S)$ of a finite number of points in $I^+(0)$. It implies in particular that it is a gluing of compact convex Euclidean polygons [Fil13]. Here also, the Fuchsian immersion is unique (among polyhedral Fuchsian convex immersions) up to global isometries. Uniqueness is not known in the general case of Theorem 1.3.

The proof of Theorem 1.3 will be by a classical polyhedral approximation, using Theorem 1.5. Hence we will need to prove some convergence and compactness results. Note that in Minkowski space, things may behave very differently than in the classical Euclidean space. Major differences may be summarized as follows.

- There is no Busemann–Feller lemma in Minkowski space. This lemma says that the orthogonal projection onto a convex set does not increase the lengths.
- The preceding fact is a consequence of the fact that there is no triangle inequality in Minkowski space. Instead, the reversed Cauchy–Schwarz inequality holds: for any vectors u and v in a Minkowski plane

$$\langle u, v \rangle_-^2 \geq \langle u, u \rangle_- \langle v, v \rangle_- .$$

If moreover u and v are spacelike, then $\langle u, u \rangle_-$ is positive, and the inequality above leads to the reversed triangle inequality

$$(4) \quad \|u\|_- + \|v\|_- \leq \|u + v\|_- ,$$

i.e. if x, y, z are three points in a Minkowski plane, related by space-like segments, then the Minkowski distance between x and y is greater than the sum of the distance between x and z and the distance between z and y .

- There is no Blaschke selection theorem. This result says that if a sequence of convex surfaces passes through some common point, it suffices to have an uniform bound on the diameters of the distance induced by the ambient Euclidean metric to have a converging subsequence of surfaces.
- The length structure given by (1) induces a distance d , which itself gives a length structure L_d . It is not straightforward that both length structure coincide on the set of Lipschitz curves. Also note that a priori, d is only a pseudo-distance (i.e. the distance of two distinct points may be zero).

Concerning the first point, there is a kind of analogue in Minkowski space, that roughly says that the orthogonal projection from the *past* of the convex set (which may be an empty set) onto the convex set expands the lengths, see [BBZ11, 6.1] for a precise statement. However in the present paper we will need only a trivial case (Lemma 3.5).

Another issue that appears in our case, is that we are looking at surfaces in Minkowski space which are invariant under the action of a group of isometries, and, given a sequence of equivariant immersions, the groups and the immersions may both degenerate, in such a way that the sequence of induced distances converges. Figure 1 shows this situation in the case of the Minkowski plane: there is an isometry I_t of Minkowski plane, corresponding to a hyperbolic translation of length t , and points ϵx and $\epsilon I_t(x)$ at (Minkowski) distance ϵ from the origin. Then it may happen that $t \rightarrow \infty$ and $\epsilon \rightarrow 0$, but the Minkowski length of the spacelike segment between ϵx and $\epsilon I_t(x)$ remains constant.

In the setting of Theorem 1.3, as we are dealing with groups acting cocompactly on \mathbb{H}^2 , the lengths of all the hyperbolic translations cannot be arbitrarily large. This point is formalized in Section 4.1.

The paper is organized as follows. In Section 2, we recall basic facts about uniform convergence of metric spaces, as well as a Theorem of A.D. Alexandrov and V.A. Zalgaller about triangulation of surfaces. In Section 3, roughly speaking, we prove that if the images of a sequence of polyhedral Fuchsian convex isometric immersions is contained between two hyperboloids, then there is a converging subsequence of immersions, and the induced distances are converging too. In Section 4 we show that if the induced distances of a sequence of polyhedral Fuchsian convex isometric immersions converge, then there is a subsequence of converging surfaces. Section 4 is the main part of the paper. Eventually, in Section 5, all the elements are put together to provide a proof of Theorem 1.3.

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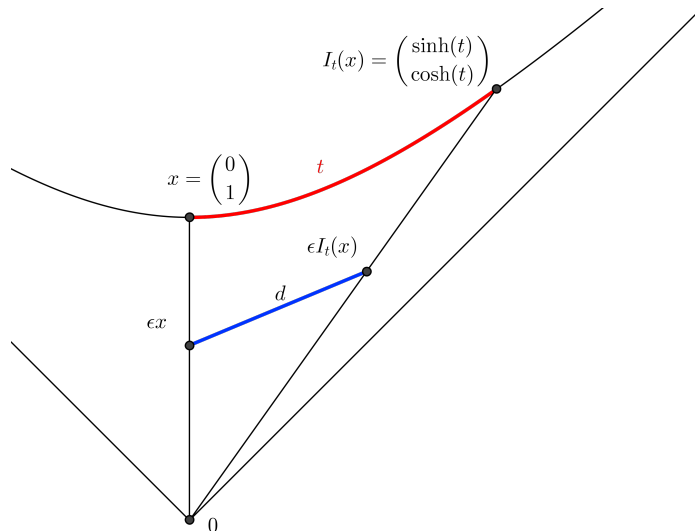


FIGURE 1. We easily compute that $\epsilon (\cosh(t) - 1)^{1/2} = d/\sqrt{2}$, so there is a suitable ϵ such that even if t is arbitrary large or close to 0, the length d between ϵx and $\epsilon I_t(x)$ remains constant, with I_t the isometry of the Minkowski plane represented by $\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$.

2. UNIFORM CONVERGENCE OF METRIC SPACES

We first recall some very basic facts about uniform convergence, for which we were unable to find a reference. Then in Section 2.3 we recall a theorem of Alexandrov and Zalgaller about polyhedral approximations of particular distances on surfaces, which is the cornerstone of the proof of our main theorem.

In the sequel, we will denote $\mathbb{N} \cup \{\infty\}$ by $\bar{\mathbb{N}}$. When we will say “ d is a distance on the manifold M ”, we imply that the topology induced by d is the topology of the manifold M .

2.1. Length convergence. Let \mathfrak{L} be a length structure on a (connected) manifold M . The length structure \mathfrak{L} sends curves of a given set of curves (supposed non empty), the set of *admissible curves*, to \mathbb{R} . The pseudo-distance d induced by \mathfrak{L} is defined as follows: $d(x, y)$ is the infimum of the lengths of admissible curves between x and y .

In turn, the distance d itself induces a length structure, denoted by L_d , and defined as follows: the length of a curve $c : [a, b] \rightarrow M$ is defined as

$$(5) \quad L_d(c) = \sup_{\delta} \sum_{i=1}^n d(c(t_i), c(t_{i+1}))$$

where the sup is taken over all the decompositions

$$\delta = \{(t_1, \dots, t_n) | t_1 = a \leq t_2 \leq \dots \leq t_n = b\}.$$

A curve is *rectifiable* if its L_d -length is finite. The length structure L_d is lower-semicontinuous [BBI01, Proposition 2.3.4]: if a sequence of rectifiable curves $c_n : [a, b] \rightarrow M$

converges to c (i.e. $c_n(t) \rightarrow c(t)$ for all t), then

$$L_d(c) \leq \liminf_n L_d(c_n) .$$

In general, on the set of admissible curves (for \mathfrak{L}),

$$L_d \leq \mathfrak{L}$$

unless \mathfrak{L} is lower-semicontinuous, as shows the following result.

Proposition 2.1 ([BBI01, Theorem 2.4.3]). *On the set of admissible curves, $L_d = \mathfrak{L}$ if and only if \mathfrak{L} is lower-semicontinuous.*

In the other way, starting from a distance d on M , it induces a length structure L_d , and this one induces a metric \hat{d} on M . The distance d is called *intrinsic* if $\hat{d} = d$, i.e. $d(x, y)$ is the inf of the L_d -length of rectifiable curves between x and y . If d comes from a length structure \mathcal{L} , then d is intrinsic [BBI01, Proposition 2.4.1].

Lemma 2.2. *Let d_n , $n \in \mathbb{N}$ be intrinsic distances on a manifold M , and let \mathfrak{L}_∞ be a length structure inducing a pseudo-distance d_∞ on M . Suppose that for any \mathfrak{L}_∞ -admissible curve c*

$$L_{d_n}(c) \rightarrow \mathfrak{L}_\infty(c) .$$

Then for any $x, y \in M$,

$$\limsup_{n \rightarrow \infty} d_n(x, y) \leq d_\infty(x, y) .$$

Proof. As d_∞ is an infimum of length of curves, for every $\epsilon > 0$ there exists a curve c on M connecting x and y such that

$$(6) \quad \mathfrak{L}_\infty(c) < d_\infty(x, y) + \epsilon .$$

For any n , $d_n(x, y) \leq L_{d_n}(c)$, then, together with the assumption of the lemma:

$$\limsup_{n \rightarrow \infty} d_n(x, y) \leq \limsup_{n \rightarrow \infty} L_{d_n}(c) = \mathfrak{L}_\infty(c) .$$

By (6), we get

$$\limsup_{n \rightarrow \infty} d_n(x, y) < d_\infty(x, y) + \epsilon .$$

Since ϵ is arbitrary, the conclusion holds. \square

Recall that a sequence of metric spaces $(M_n, d_n)_n$ uniformly converge to the metric space (M, d) if there exists homeomorphisms $f_n : M_n \rightarrow M$ such that

$$\sup_{x, y \in M_n} |d(f_n(x), f_n(y)) - d_n(x, y)|$$

goes to 0 when n goes to infinity. If $M_n = M$ for all n , then the definition is the usual definition of uniform convergence, considering the distances as maps from $M \times M$ to \mathbb{R} . This is a more restrictive notion of convergence than the usual Gromov–Hausdorff convergence. Actually, uniform convergence implies Gromov–Hausdorff convergence, see e.g. [BBI01]. But uniform convergence is the suitable notion for our needs.

The following trick is maybe due to Alexandrov [Ale06].

Lemma 2.3. *Suppose that distances (d_n) pointwise converge to d_∞ on a compact manifold M , and that there is a distance d_{\max} such that for any $n \in \mathbb{N}$, $d_n \leq d_{\max}$. Then the convergence is uniform.*

Proof. By the triangle inequality,

$$|d_n(x, y) - d_n(p, q)| \leq d_n(x, p) + d_n(y, q)$$

$$\leq d_{\max}(x, p) + d_{\max}(y, q) .$$

So the family of continuous functions d_n on $M \times M$ endowed with the product distance $d_{\max} + d_{\max}$ is equi-Lipschitz, hence equicontinuous. By Arzela–Ascoli theorem, the convergence of $(d_n)_n$ is uniform. \square

2.2. Convergence on the universal cover. Let us choose a point x_o in the manifold M . We will denote by $\pi_1 M$ the fundamental group of M based at x_o . Let \tilde{M} be the universal cover of M , on which $\pi_1 M$ acts by deck transformations. Let \mathbf{p} be the projection $\tilde{M} \rightarrow M$.

A distance d on M defines a length structure $\tilde{\mathfrak{L}}$ on the set of lifts of rectifiable curves on M : the length of c on \tilde{M} is defined as the L_d -length of $\mathbf{p}(c)$ on M . The length structure $\tilde{\mathfrak{L}}$ defines a metric \tilde{d} on \tilde{M} . As L_d is lower semicontinuous, \tilde{L} is lower semicontinuous, and by Proposition 2.1, $\tilde{L} = L_{\tilde{d}}$.

Recall the following classical result.

Theorem 2.4 (Hopf–Rinow, [BH99, I.3.7], [BBI01, 2.5.23]). *If an intrinsic distance d on M is complete, then for any $x, y \in M$, there exists a continuous curve joining x and y whose length is equal to $d(x, y)$.*

Such a curve is called a *shortest path*. In particular, a shortest path is rectifiable. Of course, if d is given by a length structure \mathfrak{L} , there is no reason why a shortest path should be admissible for \mathfrak{L} .

Let us suppose that M is compact, and that d is an intrinsic distance. The metric space (\tilde{M}, \tilde{d}) is complete, hence Hopf–Rinow theorem applies, see [BH99, I.8.3(2)–8.4(1)].

Lemma 2.5. *Let M be a compact manifold, and d_n, d_∞ distances on M , such that d_n uniformly converge to d_∞ . The map $d_{\sup} : M \times M \rightarrow \mathbb{R}$ defined by*

$$d_{\sup}(x, y) = \sup_{n \in \mathbb{N}} d_n(x, y)$$

is a distance on M .

Actually it is straightforward that d_{\sup} is a distance. What we imply in the statement of this lemma is that the topology induced by d_{\sup} is the same as the topology of M . It relies on the following fact.

Fact 2.6. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of continuous functions on a compact metric space (E, m) , uniformly converging to a function f_∞ . Then the function $g = \sup_{n \in \mathbb{N}} f_n$ is continuous.*

Proof. Let $\epsilon > 0$. The function f_∞ is continuous hence uniformly continuous on E . So there exists $\delta_\infty > 0$ such that if $m(x, y) < \delta_\infty$, then

$$|f_\infty(x) - f_\infty(y)| \leq \epsilon/3 .$$

Moreover, by uniform convergence, there exists N such that for any $n \geq N$ and any $x \in E$,

$$|f_n(x) - f_\infty(x)| \leq \epsilon/3$$

so for $n \geq N$ and x, y such that $m(x, y) < \delta_\infty$,

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_\infty(x)| + |f_\infty(x) - f_\infty(y)| + |f_\infty(y) - f_n(y)| \leq \epsilon .$$

Also, for any $n \leq N$, the function f_n is uniformly continuous, hence there exists $\delta_n > 0$ such that if $m(x, y) < \delta_n$ then

$$|f_n(x) - f_n(y)| \leq \epsilon .$$

So, for $\delta = \min\{\delta_\infty, \delta_n, n \leq N\}$, if $m(x, y) < \delta$, then $|f_n(x) - f_n(y)| < \epsilon$, for any $n \in \tilde{\mathbb{N}}$. In particular, for all $n \in \tilde{\mathbb{N}}$

$$f_n(y) < f_n(x) + \epsilon \leq g(x) + \epsilon ,$$

which implies that $g(y) \leq g(x) + \epsilon$. Similarly, $g(x) \leq g(y) + \epsilon$. \square

We will denote by $B_m(x, r)$ the open ball of center x and radius r for the distance m .

Proof of Lemma 2.5. Let O be an open set of M . For any $x \in O$, there is an $\epsilon > 0$ such that $B_{d_\infty}(x, \epsilon) \subset O$. But $B_{d_{\text{sup}}}(x, \epsilon) \subset B_{d_\infty}(x, \epsilon)$, so O is an open set for the topology induced by d_{sup} .

Let $\epsilon > 0$. Let \bar{B} be a closed ball centred at x with radius ϵ for d_{sup} . By Fact 2.6, $d_{\text{sup}}^{-1}([0, \epsilon])$ is a closed set of $M \times M$. As \bar{B} is the projection onto the second factor of $d_{\text{sup}}^{-1}([0, \epsilon]) \cap (\{x\} \times M)$, it is a closed set of M (by the tube lemma, as M is compact, then the projection is a closed map). \square

Note that by the definition of the length of the curve and by the definition of the metric on the universal cover, with the assumptions of Lemma 2.5, for any $n \in \tilde{\mathbb{N}}$,

$$\tilde{d}_n \leq \tilde{d}_{\text{sup}} .$$

Lemma 2.7. *Let M be a compact manifold, and d_n, d_∞ distances on M , such that d_n uniformly converge to d_∞ . For any $x \in M$ there exists an open set U with $x \in U$ such that the restriction to U of the projection $\mathbf{p} : \tilde{M} \rightarrow M$ is an isometry for all the distances $d_n, n \in \tilde{\mathbb{N}}$, if n is sufficiently large.*

Proof. Let O be an open set of \tilde{M} such that $\mathbf{p}|_O : O \rightarrow \mathbf{p}(O)$ is a homeomorphism and let s be the inverse of $\mathbf{p}|_O$. Then for any r_n such that $B_{d_n}(x, 2r_n) \subset \mathbf{p}(O)$, s is an isometry on $B_{d_n}(x, r_n)$ for d_n , see the proof of Proposition I-3.25 in [BH99].

Let R be such that $B_{d_\infty}(x, 2R) \subset \mathbf{p}(O)$. Let $\epsilon > 0$ with $\epsilon < R$. Then by uniform convergence of $(d_n)_n$, there exists $N(\epsilon) \in \mathbb{N}$ such that for any integer $n > N(\epsilon)$, $B_{d_n}(x, 2(R - \epsilon)) \subset B_{d_\infty}(x, 2R)$. Thus, for every $n > N(\epsilon)$, \mathbf{p} is an isometry on $B_{d_n}(x, R - \epsilon)$.

By Lemma 2.5, $B_{d_{\text{sup}}}(x, R - \epsilon)$ is an open set, and $B_{d_{\text{sup}}}(x, R - \epsilon) \subset B_{d_n}(x, R - \epsilon)$ for any $n \in \tilde{\mathbb{N}}$. So we can take $U = B_{d_{\text{sup}}}(x, R - \epsilon)$. \square

Lemma 2.8. *Let $x, y \in \tilde{M}$. Under the hypothesis of Lemma 2.7, for any $\epsilon > 0$, for n sufficiently large,*

$$(7) \quad \tilde{d}_n(x, y) \leq \tilde{d}_\infty(x, y) + \epsilon ,$$

in particular, $\limsup_n \tilde{d}_n(x, y) \leq \tilde{d}_\infty(x, y)$.

Proof. For convenience, let us write d for d_∞ . Let $\tilde{c} : [0, 1] \rightarrow \tilde{M}$ be a shortest path for the distance \tilde{d} between x and y . By Lemma 2.7, for any $p \in \tilde{M}$, there exists an open set U_p such that, up to a finite number of indices n , \mathbf{p} is an isometry on U_p . Thus, there is an integer k and points $p_i, i = 1, \dots, k$ on the compact set $\tilde{c}([0, 1])$ such that the U_{p_i} cover $\tilde{c}([0, 1])$, $p_1 = x, p_k = y$. Let us also take some points $z_i, i = 1, \dots, k + 1$ on $\tilde{c}([0, 1])$ such that $z_1 = x, z_{k+1} = y$ and $z_i \in U_{p_i} \cap U_{p_{i+1}}$ for $1 < i < k$. Then

$$(8) \quad \tilde{d}(x, y) = L_{\tilde{d}}(\tilde{c}) \geq \sum_{i=1}^k \tilde{d}(z_i, z_{i+1}) .$$

Let $\epsilon > 0$. By uniform convergence, for n sufficiently large,

$$d_n(\mathbf{p}(z_i), \mathbf{p}(z_{i+1})) \leq d(\mathbf{p}(z_i), \mathbf{p}(z_{i+1})) + \epsilon/k$$

and as \mathbf{p} is a local isometry for all the d_n , for n sufficiently large, on any U_i (Lemma 2.7), we obtain

$$(9) \quad \tilde{d}_n(z_i, z_{i+1}) \leq \tilde{d}(z_i, z_{i+1}) + \epsilon/k .$$

Combining the formulas (8) and (9), and applying several times the triangle inequality, we get (7). \square

The following lemma is Theorem 1 and Theorem 2 p.225 of [AZ67].

Lemma 2.9. *Let M be a compact manifold, and d_n, d_∞ distances on M , such that d_n uniformly converge to d_∞ . Let c_n be curves on M . If there exists $L > 0$ such that for any n , $L_{d_n}(c_n) < L$, then, up to extract a subsequence, the curves c_n converge to a curve c , and*

$$L_{d_\infty}(c) \leq \liminf_n L_{d_n}(c_n) .$$

Proposition 2.10. *Let M be a compact manifold, and d_n, d_∞ distances on M , such that d_n uniformly converge to d_∞ . Up to extract a subsequence, $(\tilde{d}_n)_n$ converge to \tilde{d}_∞ , uniformly on compact sets.*

Proof. Let $x, y \in \tilde{M}$, and \tilde{c}_n be a shortest path between x and y for \tilde{d}_n , $n \in \mathbb{N}$. So, if $c_n = \mathbf{p}(\tilde{c}_n)$, then

$$\tilde{d}_n(x, y) = L_{\tilde{d}_n}(\tilde{c}_n) = L_{d_n}(c_n) .$$

By Lemma 2.8, $L_{d_n}(c_n)$ are uniformly bounded from above for n sufficiently large. By Lemma 2.9, up to extract a subsequence, there is a curve c between x and y with

$$\tilde{d}_\infty(x, y) \leq L_{d_\infty}(c) \leq \liminf_n L_{d_n}(c_n) = \liminf_n \tilde{d}_n(x, y) .$$

This and Lemma 2.8 gives the pointwise convergence. The result follows from Lemma 2.3. \square

Lemma 2.11. *Let d be a distance on the compact manifold M . There exists $l > 0$ such that for any $\gamma \in (\pi_1 M \setminus \{0\})$, for any $x \in \tilde{M}$, $\tilde{d}(x, \gamma.x) \geq l$.*

Proof. The proof is formally the same as the one of Lemma 1 p. 237 in [Rat06]. \square

Fact 2.12. *Let $(f_n)_n$ be a sequence of continuous functions on a compact metric space (E, m) , uniformly converging to a function f_∞ . Then the function $h = \inf_{n \in \mathbb{N}} f_n$ is continuous. Moreover, if $f_n > 0$ for $n \in \mathbb{N}$, then $h > 0$.*

Proof. The first assumption follows from Fact 2.6, because $\inf(f_n) = -\sup(-f_n)$. The second assumption is clear as for any x , $f_n(x)$ converge to $f_\infty(x)$ which is positive. \square

Lemma 2.13. *Let M be a compact manifold, and d_n, d_∞ distances on M , such that d_n uniformly converge to d_∞ , and let $d_{\sup}(x, y) = \sup_{n \in \mathbb{N}} d_n(x, y)$. There exists $c > 0$ such that for any $n \in \mathbb{N}$, $cd_{\sup} \leq d_n$.*

Proof. Let $h = \inf_n d_n$. By Fact 2.12, it is a positive continuous function, so the function h/d_{\sup} is positive and continuous on the compact set $M \times M$, and its minimum value c satisfies the lemma. \square

Combining Lemma 2.13 and Lemma 2.11 leads to the following.

Corollary 2.14. *There exists $G > 0$ such that, for every element $\gamma \in (\pi_1 M \setminus \{0\})$, for any $x \in \tilde{M}$, for any n :*

$$\tilde{d}_n(x, \gamma.x) \geq G .$$

2.3. Approximation by polyhedral metrics. Recall (see e.g. [BH99, BBI01, AKP17]) that for any triple of points (x, y, z) in an intrinsic metric space (M, m) , a *comparison triangle* is a triangle on the Euclidean plane with vertices (x', y', z') such that $m(x, y) = d_{\mathbb{R}^2}(x', y')$, $m(y, z) = d_{\mathbb{R}^2}(y', z')$, and $m(x, z) = d_{\mathbb{R}^2}(x', z')$. Using comparison triangles, it is possible to define a notion of *upper angles* between two geodesic paths in (M, m) starting from the same point (see I.1.12 in [BH99]). Following Proposition II.1.7 in [BH99] (parts (1) and (4)), we give a definition of CAT(0) space in the form which is convenient for us.

Definition 2.15. *A complete intrinsic metric space (M, m) is CAT(0) if the upper angle between any couple of sides of every geodesic triangle with distinct vertices is no greater than the angle between the corresponding sides of its comparison triangle in \mathbb{R}^2 .*

The CAT(0) condition implies that the shortest path between two points of M is unique [BH99, II 1.4].

Definition 2.16. *A complete intrinsic metric space (M, m) is of non-positive curvature (in the Alexandrov sense), if for any x there is r such that $B_m(x, r)$ endowed with the induced (intrinsic) distance is CAT(0).*

Note that a compact metric space cannot be CAT(0) as this condition implies that the space must be simply connected [BH99, II-1.5]. But by the Cartan–Hadamard theorem, if M is a manifold and (M, m) is of non-positive curvature, then (\tilde{M}, \tilde{m}) is CAT(0) [BH99, II.4.1].

We will now consider that $M = S$ is a compact surface. Recall from the introduction that a polyhedral metric of non-positive curvature on S is a flat metric on S with conical singularities of negative curvature. They admit a geodesic triangulation whose vertices are exactly the singular points [Tro86], so equivalently they can be defined as a gluing of flat triangles along isometric edges, such that the sum of the angles of triangles around each vertex is $> 2\pi$. It follows from [BH99, Lemma II.5.6] that polyhedral metrics of non-positive curvature are metrics of non-positive curvature in the sense of Definition 2.16.

Let us remind the notion of *bounded integral curvature* (or just of *bounded curvature* in terms of [AZ67, Chapter I, p. 6]).

Definition 2.17. *An intrinsic distance m on a surface S is said to be of bounded integral curvature (in short, BIC) if (S, m) verifies the following property:*

- For every $x \in S$ and every neighbourhood N_x of x homeomorphic to the open disc, for any finite system $\{T\}$ of pairwise nonoverlapping *simple* triangles T belonging to N_x , the sum of the *excesses*

$$\delta(T) = \bar{\alpha}_T + \bar{\beta}_T + \bar{\gamma}_T - \pi$$

of the triangles $T \in \{T\}$ with upper angles $(\bar{\alpha}_T, \bar{\beta}_T, \bar{\gamma}_T)$, is bounded from above by a number C depending only on the neighbourhood N_x , i.e.

$$\sum_{T \in \{T\}} \delta(T) \leq C .$$

A simple triangle is a triangle bounding an open set homeomorphic to a disc, and which is *convex relative to the boundary*, i.e. no two points of the boundary of the triangle can be joined by a curve outside the triangle which is shorter than a suitable part of the boundary joining the points, see [AZ67] for more details.

Lemma 2.18. *A metric of non-positive curvature on a compact surface is BIC.*

Proof. By definition, each point x of a metric of non-positive curvature on a compact surface space has a CAT(0) neighbourhood N_x . Therefore, by Definition 2.15, for any geodesic triangle $T \subset N_x$, the angles of the comparison triangle $\bar{T} \subset \mathbb{R}^2$ are not less than the corresponding upper angles of T . As the sum of the angles of a Euclidean triangle is π , the excess of the triangle T (see Definition 2.17) is

$$\delta(T) \leq \delta(\bar{T}) = 0.$$

Hence, for any finite system $\{T\}$ of pairwise nonoverlapping simple triangles T in N_x , the sum of the excesses is trivially non-positive. \square

We want to find a sequence d_n of polyhedral metrics of non-positive curvature on a compact surface S converging to a given distance d of non-positive curvature on S .

The main tool is Theorem 10 in [AZ67, Chapter III, p. 84]. This result adapted to our case is formulated as follows. We call a *flat metric* on a surface a distance such that each point has a neighbourhood isometric to a Euclidean cone. The cone has arbitrary angle at the vertex. If the angle is 2π , the point is regular, and singular otherwise. If the angles at the singular points are $> 2\pi$, a flat metric is a polyhedral metric of non-positive curvature.

Theorem 2.19. *Given a compact BIC surface, there is a sequence of flat metrics converging uniformly to it.*

We will also need the following technical result, which corresponds to Theorem 11 in [AZ67, Chapter II, p. 47]. Here we write it down in a convenient form.

Lemma 2.20. *Let p be a point on a BIC surface such that there is at least one shortest arc containing p in its interior. Then for any decomposition of a neighbourhood of p into sectors convex relative to the boundary formed by geodesic rays issued from p such that the upper angles between the sides of these sectors exist and do not exceed π , the total sum of those angles is not less than 2π .*

Corollary 2.21. *Let d be a distance of non-positive curvature on a compact surface S . Then there exists a sequence of polyhedral metrics of non-positive curvature $(d_n)_{n \in \mathbb{N}}$ on S which uniformly converges to d .*

Proof. Applying Theorem 2.19, we obtain a sequence $(d_n)_{n \in \mathbb{N}}$, of flat metrics converging uniformly to d . We have to check that the total angles around the conical singularities of d_n , $n \in \mathbb{N}$, are not less than 2π .

In the proof of Theorem 2.19 the distances d_n , $n \in \mathbb{N}$, are constructed as follows:

1. construct a geodesic triangulation τ_n of (S, d) ;
2. replace the interiors of the triangles of τ_n by the interiors of the Euclidean comparison triangles.

Note that point 1. is far from being trivial. Then, one has to prove that the finer the triangulation is, the closer d_n is from d (for the uniform distance between metric spaces).

Remark that, by construction (see the proof of Theorem 10 in [AZ67, Chapter III, p. 85, lines 3 and 4]), every vertex of τ_n lies in the interior of some geodesic in (S, d) (this also follows because (S, d) is of non-positive curvature [BH99, II.5.12]). Applying Lemma 2.20, we immediately get that the sum of the sector angles at any vertex V of a triangulation τ_n in (S, d) , is not less than 2π . By Definition 2.15, the angles of the comparison triangles in \mathbb{R}^2 are not less than the corresponding sector angles at V of the triangulation τ_n in (S, d) . Hence the total angle around every singular point of the polyhedral metric d_n is not less than 2π . \square

3. CONVERGENCE OF INDUCED DISTANCES

The aim of this section is to prove Proposition 3.22.

3.1. Graphs on the hyperboloid. Let K be a spacelike convex set, and suppose that its boundary G_u is the graph of a positive function u on \mathbb{H}^2 , i.e.

$$(10) \quad G_u = \{u(x)x \mid x \in \mathbb{H}^2\}.$$

Note that $G_u \subset I^+(0)$.

Definition 3.1. A function $u : \mathbb{H}^2 \rightarrow \mathbb{R}$ is H -convex if G_u defined by (10) is a spacelike convex surface.

The function $u : \mathbb{H}^2 \rightarrow \mathbb{R}$ can be written as

$$u(x) = \inf\{\lambda \geq 0 \mid \lambda x \in K\}$$

so it is the restriction to \mathbb{H}^2 of the function $U : I^+(0) \rightarrow \mathbb{R}$ defined by

$$U(x) = \inf\{\lambda \geq 0 \mid \lambda x \in K\}.$$

Let $\mu \geq 0$. As $U(x)x \in \partial K$, then $\frac{U(x)}{\mu}\mu x \in \partial K$. Also, $U(\mu x)\mu x \in \partial K$. As for any $y \in I^+(0)$, there exists exactly one $\lambda > 0$ such that $\lambda y \in \partial K$, it follows that $U(\mu x) = \frac{U(x)}{\mu}$, i.e. U is (-1) -homogeneous.

Also, if $x \in K$ and $\lambda \geq 1$, then $\lambda x \in K$, so $x \in \partial K$ if and only if $U(x) = 1$. If $x \in K$, there exists $\lambda \leq 1$ such that $\lambda x \in \partial K$. In particular, if $x \in K$ and $\lambda \leq 1$, then $U(\lambda x) = 1$ i.e. $U(x) = \lambda \leq 1$, so

$$K = \{x \mid U(x) \leq 1\}.$$

Lemma 3.2. Let u be a H -convex function, and let U be its (-1) -homogeneous extension on $I^+(0)$. The function $-\frac{1}{U}$ is convex.

Proof. By definition of U ,

$$(11) \quad \frac{1}{U}(z) = \max\{t \geq 0 \mid z \in tK\}.$$

Let $x, y \in I^+(0)$. Hence $x \in \frac{1}{U}(x)K$ and $y \in \frac{1}{U}(y)K$, so, as K is convex, for any λ, μ , $\lambda K + \mu K = (\lambda + \mu)K$ [Sch14, Remark 1.1.1], hence, for $0 \leq a \leq 1$,

$$ax + (1-a)y \in \left(a \frac{1}{U}(x) + (1-a) \frac{1}{U}(y)\right) K$$

and by (11),

$$a \frac{1}{U}(x) + (1-a) \frac{1}{U}(y) \leq \frac{1}{U}(ax + (1-a)y)$$

hence $\frac{1}{U}$ is concave. \square

We will use the strong fact that the function $-\frac{1}{U}$ is convex.

Lemma 3.3. *Let $(u_n)_n$ be a sequence of H -convex functions, such that there exists $\beta > \alpha > 0$ with $\alpha < u_n < \beta$. Up to extract a subsequence, $(u_n)_n$ converges to a H -function u , uniformly on compact sets.*

Proof. Let $X \in W \subset I^+(0)$, where W is a compact set that does not touch $\partial I^+(0)$. By (-1) -homogeneity of the U_n ,

$$U_n(X) = (-\langle X, X \rangle_-)^{-1/2} u_n \left(\frac{X}{(-\langle X, X \rangle_-)^{1/2}} \right),$$

so as for any $x \in \mathbb{H}^2$, the sequence $(u_n(x))_n$ is bounded, and as $W \cap \partial I^+(0) = \emptyset$, the sequence $(-\frac{1}{U_n}(x))_n$ is bounded, and by standard property of convex functions, [Roc97, 10.9], there is a function U such that, up to extract a subsequence, $(-\frac{1}{U_n})_n$ converges uniformly on each compact subsets of the interior of W to a convex function $-\frac{1}{U}$. The set

$$K = \{x \mid -\frac{1}{U(x)} \leq -1\}$$

is convex, and, as U is clearly (-1) -homogeneous,

$$\partial K = \{x \mid -\frac{1}{U(x)} = -1\} = \{x \mid U(x) = 1\}.$$

is a convex surface, and it is the graph of the restriction u of U on \mathbb{H}^2 .

Support planes of ∂K are limits of support planes of ∂K_n , hence spacelike or lightlike. Let us suppose that ∂K has a lightlike support plane P . Then ∂K is on one side of P , but meets all the hyperboloids centred at zero. This contradicts the fact that ∂K is a graph above such an hyperboloid. \square

Lemma 3.4. *The functions u_n and u of Lemma 3.3 are equi-Lipschitz on any compact set of \mathbb{H}^2 .*

Proof. If U_n and U are the (-1) homogeneous extensions of u_n and u respectively, we know by Lemma 3.2 that $-1/U_n$ and $-1/U$ are convex.

As $H_n = -1/U_n$ is convex on $I^+(0)$, then, for a compact set $C \subset \mathbb{H}^2$, there exists a $\epsilon > 0$ such that for all $x, y \in C$, [Roc97, 10.4] (here $\|\cdot\|$ is the Euclidean norm on \mathbb{R}^3)

$$|H_n(x) - H_n(y)| \leq \frac{\max_C H_n - \min_C H_n}{\epsilon} \|x - y\|$$

that leads to

$$|u_n(x) - u_n(y)| \leq \frac{\max u_n}{\epsilon} \left(\frac{\max u_n}{\min u_n} - 1 \right) \|x - y\|$$

and as $\alpha < u_n < \beta$, the $(u_n)_n$ are equi-Lipschitz on $C \cap \mathbb{H}^2$, for the distance on \mathbb{H}^2 induced by the ambient Euclidean one. But all the norms coming from Riemannian structures are locally equivalent on \mathbb{H}^2 . \square

3.2. Convergence of the length structures. This part is a straightforward adaptation of classical results from the Euclidean setting [Ale06]. Let u be a H -convex function. Let $c : [a, b] \rightarrow \mathbb{H}^2$ be a Lipschitz curve. Let $v = (u \circ c)$. When the derivative exists,

$$v' = (u \circ c)'c + u(c)c'$$

but $c'(s)$ belongs to $T_{c(s)}\mathbb{H}^2$, which is orthogonal to $c(s)$, and $\langle c(s), c(s) \rangle_- = -1$, and

$$\langle c'(s), c'(s) \rangle_-^{1/2} = \|c'\|_{\mathbb{H}^2}$$

is the norm induced by the hyperbolic metric. So

$$\langle v', v' \rangle_- = u^2(c) \|c'\|_{\mathbb{H}^2}^2 - ((u \circ c)')^2$$

and we define the following length

$$(12) \quad \mathfrak{L}_u(c) = \int_a^b (u^2(c) \|c'\|_{\mathbb{H}^2}^2 - ((u \circ c)')^2)^{1/2} .$$

Note that as \mathbb{H}^2 is a smooth hypersurface, $\mathcal{L}_1 = L_{d_{\mathbb{H}^2}}$ on the set of Lipschitz curves, see e.g. [Bur15]. The following is immediate.

Lemma 3.5. *Let u be a H -convex function and $\beta > 0$ with $u < \beta$. For any Lipschitz curve c on \mathbb{H}^2 ,*

$$\mathfrak{L}_u(c) \leq \beta L_{d_{\mathbb{H}^2}}(c) .$$

Let u_n be H -convex functions converging to a H -convex function u . As the u_n are Lipschitz on \mathbb{H}^2 , by Rademacher theorem, they are differentiable almost everywhere (for the Borel measure given by the hyperbolic metric). As there is a countable number of u_n , there exists a set $\mathcal{D} \subset \mathbb{H}^2$ of zero measure, such that the u_n and u are differentiable on $\mathbb{H}^2 \setminus \mathcal{D}$.

Let $c : I \rightarrow \mathbb{H}^2$ be a Lipschitz curve. The subset $c^{-1}(\mathcal{D})$ may be a set of non-zero measure in I . However, $u \circ c$ is a Lipschitz function on I , hence it is derivable almost everywhere on I , and moreover, all the $u_n \circ c$ are simultaneously derivable almost everywhere. To illustrate, let us consider the example of the intersection of two halfplanes. It is the graph of a function f , which is differentiable everywhere except on the projection c of the edge onto the plane. But the restriction of f to c is derivable everywhere on c .

Lemma 3.6. *Let u_n be H -convex functions converging to a H -convex function u . For almost all $t \in I$, u_n and u are derivable and at such a point t , up to extract a subsequence, $u_n(c(t))' \rightarrow u(c(t))'$.*

Proof. Let us denote by X the unit timelike vector $c(t)$ and by Y the unit spacelike vector $c'(t)$. As c is a curve on \mathbb{H}^2 , $\langle X, Y \rangle_- = 0$. The tangent vector of the curve $(u_n \circ c)c$ is

$$V_n = (u_n(c(t)))'X + u_n(c(t))Y$$

and in the plane P spanned by X and Y , the vector

$$N_n = u_n(c(t))X + (u_n(c(t)))'Y$$

is orthogonal to V_n for $\langle \cdot, \cdot \rangle_-$. It follows from Lemma 3.4 and the fact that the u_n are uniformly bounded, that the Euclidean norms of N_n are uniformly bounded. Hence, up to extract a subsequence, $(N_n)_n$ converges to a vector N . Note that N is non-zero, otherwise $\langle N_n, X \rangle_- = -u_n(c(t))$ would converge to 0, that is impossible.

Let A_n be the intersection of the convex set K_n defined by u_n and the plane P . The set A_n is a convex set, and V_n is a tangent vector, hence, for any $y \in \mathbb{H}^2 \cap P$,

$$\langle N_n, u_n(c(t))X - u_n(y)y \rangle_- \geq 0$$

and passing to the limit,

$$\langle N, u(c(t))X - u(y)y \rangle_- \geq 0$$

that says that N is a normal vector of A (the intersection of K with P), in particular, N is orthogonal to

$$V = (u(c(t)))'X + u(c(t))Y .$$

It follows that there exists λ such that

$$\lambda N = u(c(t))X + (u(c(t)))'Y$$

but as $\langle N_n, X \rangle_-$ converges to $\langle N, X \rangle_-$ and $u_n(c(t))$ converges to $u(c(t))$, then $\lambda = 1$ and $(u_n(c(t)))'$ must converge to $(u(c(t)))'$. \square

The preceding result and the Dominated convergence theorem give the following.

Proposition 3.7. *For any Lipschitz curve $c : [a, b] \rightarrow \mathbb{R}$ on \mathbb{H}^2 , if $u_n \rightarrow u$, then, up to extract a subsequence, $\mathfrak{L}_{u_n}(c) \rightarrow \mathfrak{L}_u(c)$.*

3.3. Estimates for the induced distances. In the preceding section, we defined a length structure \mathfrak{L}_u . Let d_u be the distance defined by this length structure. To be more precise, for the moment d_u is only a pseudo-distance. As we noticed in Section 2.1, the pseudo-distance d_u is intrinsic. We have a first bound, immediate from Lemma 3.5.

Lemma 3.8. *If $u \leq \beta$, then $d_u \leq \beta d_{\mathbb{H}^2}$.*

Lemma 3.9. *Let u be a H -convex function such that d_u is a complete distance with Lipschitz shortest paths. Let $\alpha > 0$ with $u > \alpha$. Then*

$$d_{\mathbb{H}^2}(x, y) \leq \frac{1}{\alpha} d_u(x, y) + \frac{1}{\alpha^2} \int_0^{d_u(x, y)} \sqrt{\langle \nu(t), \nu'(t) \rangle_-^2} dt ,$$

where ν is an arc-length parametrized shortest path on the graph of u between $u(x)x$ and $u(y)y$.

Proof. Let $\text{pr}(y) = \frac{y}{\sqrt{-\langle y, y \rangle_-}}$ be the radial projection from the future cone of the origin of Minkowski space onto \mathbb{H}^2 . By definition of the length, we have that

$$d_{\mathbb{H}^2}(x, y) \leq L_{d_{\mathbb{H}^2}}(\text{pr} \circ \nu)$$

where

$$L_{d_{\mathbb{H}^2}}(\text{pr} \circ \nu) := \int_0^{d_u(x, y)} \|(\text{pr} \circ \nu)'(t)\|_{\mathbb{H}^2} dt = \int_0^{d_u(x, y)} \sqrt{\langle (\text{pr} \circ \nu)'(t), (\text{pr} \circ \nu)'(t) \rangle_-} dt .$$

A straightforward computation gives

$$L_{d_{\mathbb{H}^2}}(\text{pr} \circ \nu) = \int_0^{d_u(x, y)} \sqrt{\frac{\langle \nu'(t), \nu'(t) \rangle_-}{-\langle \nu(t), \nu(t) \rangle_-} + \frac{\langle \nu(t), \nu'(t) \rangle_-^2}{\langle \nu(t), \nu(t) \rangle_-^2}} dt ,$$

so, as $\sqrt{a^2 + b^2} \leq a + b$ for $a > 0$ and $b > 0$,

$$d_{\mathbb{H}^2}(x, y) \leq \int_0^{d_u(x, y)} \sqrt{\frac{\langle \nu'(t), \nu'(t) \rangle_-}{-\langle \nu(t), \nu(t) \rangle_-}} dt + \int_0^{d_u(x, y)} \sqrt{\frac{\langle \nu(t), \nu'(t) \rangle_-^2}{\langle \nu(t), \nu(t) \rangle_-^2}} dt .$$

For every $y \in \mathbb{H}^2$ we have $\langle y, y \rangle_- = -1$, also, for each t there is $x \in \mathbb{H}^2$ such that $\nu(t) = u(x)x$, and $u > \alpha$ by assumption, therefore $-\langle \nu(t), \nu(t) \rangle_- \geq \alpha^2$ for all t , and so

$$d_{\mathbb{H}^2}(x, y) \leq \frac{1}{\alpha} \int_0^{d_u(x, y)} \sqrt{\langle \nu'(t), \nu'(t) \rangle_-} dt + \frac{1}{\alpha^2} \int_0^{d_u(x, y)} \sqrt{\langle \nu(t), \nu'(t) \rangle_-^2} dt .$$

As $\int_0^{d_u(x, y)} \sqrt{\langle \nu'(t), \nu'(t) \rangle_-} dt = d_u(x, y)$, we obtain the result.

□

Lemma 3.10. *Let u be a H -convex function, with $\alpha, \beta > 0$ such that $\alpha < u < \beta$, and let ν be a path on G_u , and t such that $\nu'(t)$ exists. Then*

$$\langle \nu(t), \nu'(t) \rangle_-^2 \leq \beta^2 - \alpha^2 .$$

Proof. Let C be the closure of the cone of all the lines containing the point $\nu(t)$ which meet $\beta\mathbb{H}^2$. Let C^0 be the closure of the complement of C .

Note that $\nu'(t)$ lies in a spacelike plane P which is a support plane of G_u . As this surface is in the past of $\beta\mathbb{H}^2$, the plane P is in the past of the plane tangent to $\beta\mathbb{H}^2$ which is parallel to P . In particular, the line from $\nu(t)$ and directed by $\nu'(t)$ never crosses $\beta\mathbb{H}^2$, i.e. is in C^0 .

Let $I(\nu(t))$ be the isotropic cone at $\nu(t)$. Note that $I(\nu(t)) \subset C$.

Let Q be the Minkowski plane containing $\nu(t)$ and $\nu'(t)$, and let dS be the set of unit spacelike vectors in Q centred at $\nu(t)$, see Figure 2. The function $f = \langle \nu(t), \cdot \rangle_-$ defined on dS is increasing and goes to $+\infty$ when approaching the past component of the isotropic cone $I(\nu(t))$. So the restriction of f to $C^0 \cap dS$ attains its maximal value at a vector v of $\partial C^0 \cap Q$. In particular, $\langle \nu(t), \nu'(t) \rangle_-$ is bounded from above by the positive quantity $\langle \nu(t), v \rangle_-$.

By definition, the line from the point $\nu(t)$ directed by the vector v is tangent to $\beta\mathbb{H}^2$. Such a vector v is defined by the fact that there exists $s \in \mathbb{R}$ satisfying

$$\begin{cases} \langle \nu(t) + sv, \nu(t) + sv \rangle_- = -\beta^2 \\ \langle v, \nu(t) + sv \rangle_- = 0 \end{cases}$$

that gives $\langle \nu(t), v \rangle_-^2 = \langle \nu(t), \nu(t) \rangle_- + \beta^2$. As $\nu(t)$ is in the future of $\alpha\mathbb{H}^2$,

$$\langle \nu(t), \nu(t) \rangle_- \leq -\alpha^2 .$$

At the end,

$$\langle \nu(t), \nu'(t) \rangle_-^2 \leq \langle \nu(t), v \rangle_-^2 \leq \beta^2 - \alpha^2 .$$

□

Corollary 3.11. *Let u be a H -convex function such that d_u is a complete distance with Lipschitz shortest paths. Then d_u is bi-Lipschitz to $d_{\mathbb{H}^2}$. More precisely, if $\alpha, \beta > 0$ are such that $\alpha < u < \beta$, then*

$$(13) \quad \left(\frac{1}{\alpha} + \frac{\sqrt{\beta^2 - \alpha^2}}{\alpha^2} \right)^{-1} d_{\mathbb{H}^2}(x, y) \leq d_u(x, y) \leq \beta d_{\mathbb{H}^2} .$$

Proposition 3.12. *Let $(u_n)_n$ be a sequence of H -convex functions such that:*

- d_{u_n} is a complete distance with Lipschitz shortest paths;
- $\mathfrak{L}_{u_n} = L_{d_{u_n}}$ on the set of Lipschitz curves;
- there are $\beta > \alpha > 0$ such that $\alpha < u_n < \beta$.

Then, up to extract a subsequence, (u_n) converges to a H -convex function u , and $(d_{u_n})_n$ converges to d_u , uniformly on compact sets.

Note that by (13), d_u is a complete distance on \mathbb{H}^2 .

Proof. The existence of u follows from Lemma 3.3. Let $\epsilon > 0$, $x, y \in \mathbb{H}^2$ and $\delta = (t_1, \dots, t_m)$ a decomposition of $[0, 1]$. Let c_n be shortest paths for d_{u_n} between x and y . By Corollary 3.11, there are positive constants k_1, k_2 such that

$$L_{k_1 d_{\mathbb{H}^2}}(c_n) \leq L_{d_{u_n}}(c_n) = d_{u_n}(x, y) \leq k_2 d_{\mathbb{H}^2}(x, y)$$

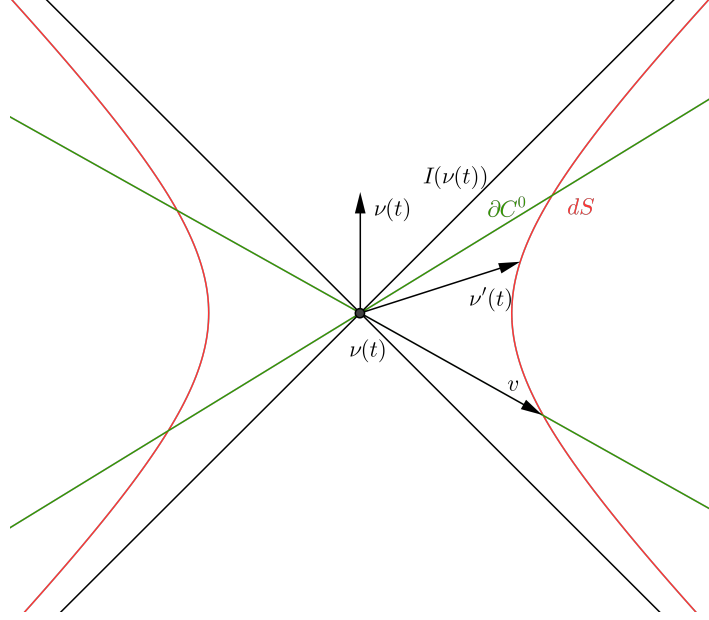


FIGURE 2. To the proof of Lemma 3.10

so, the $d_{\mathbb{H}^2}$ -length of the c_n is bounded from above independently of n . Let us also denote by c_n a reparametrization of c_n defined on $[0, 1]$ proportionally to the hyperbolic arc-length. As the endpoints of the curve are fixed, all the curves c_n are contained in a closed ball for the metric $d_{\mathbb{H}^2}$. By the assumptions, a closed ball for $d_{\mathbb{H}^2}$ is compact [BBI01, 2.5.22]. So we can apply Arzela–Ascoli theorem ([Pap05, 1.4.10], [BBI01, 2.5.14]): up to extract a subsequence, $(c_n)_n$ uniformly converges to a curve c_∞ . Note that the c_n are equi-Lipschitz, hence c_∞ is Lipschitz. So, there is N_1 such that if $n \geq N_1$ then, $\forall i \in \{1, \dots, m\}$,

$$d_{u_n}(c_n(t_i), c_\infty(t_i)) < k_2 d_{\mathbb{H}^2}(c_n(t_i), c_\infty(t_i)) < \frac{\epsilon}{3m} ,$$

i.e. $\sum_{i=1}^m d_{u_n}(c_n(t_i), c_\infty(t_i)) < \frac{\epsilon}{3}$, and triangle inequality gives

$$\sum_{i=1}^m d_{u_n}(c_\infty(t_i), c_\infty(t_{i+1})) \leq \sum_{i=1}^m d_{u_n}(c_n(t_i), c_n(t_{i+1})) + \frac{2}{3}\epsilon .$$

Taking the sup for the decomposition:

$$L_{d_{u_n}}(c_\infty) \leq L_{d_{u_n}}(c_n) + \frac{2}{3}\epsilon .$$

On the other hand, by Proposition 3.7 and by hypothesis, after taking a suitable subsequence, there is N_2 such that for $n \geq N_2$,

$$\mathfrak{L}_u(c_\infty) \leq L_{d_{u_n}}(c_\infty) + \frac{\epsilon}{3} .$$

Hence, for $n \geq \max\{N_1, N_2\}$, $\mathfrak{L}_u(c_\infty) \leq L_{d_{u_n}}(c_n) + \epsilon$, so $\mathfrak{L}_u(c_\infty) \leq \liminf_n L_{d_{u_n}}(c_n) + \epsilon$ and as ϵ is arbitrary,

$$\mathfrak{L}_u(c_\infty) \leq \liminf_n L_{d_{u_n}}(c_n) .$$

As $L_{d_{u_n}}(c_n) = d_{u_n}(x, y)$ and $d_u(x, y) \leq \mathfrak{L}_u(c_\infty)$, we obtain

$$d_u(x, y) \leq \liminf_{n \rightarrow \infty} d_{u_n}(x, y) .$$

This, together with Lemma 2.2, leads to the pointwise convergence of d_{u_n} to d_u . The local uniform convergence comes from Lemma 2.3. \square

3.4. Convergence on the quotient. This section is a straightforward adaptation of [FIV16].

The topology on the space of representations $\rho : \pi_1 S \rightarrow O_0(2, 1) \cong \mathrm{PSL}(2, \mathbb{R})$ can be defined as follows. Let us choose a set of $2g$ generators $(\gamma_1, \dots, \gamma_{2g})$ of $\pi_1 S$. One says that $(\rho_n)_n$ converges to ρ if $(\rho_n(\gamma_1), \dots, \rho_n(\gamma_{2g}))$ converges to $(\rho(\gamma_1), \dots, \rho(\gamma_{2g}))$ in $\mathrm{PSL}(2, \mathbb{R})^{2g} \subset (\mathbb{R}^4)^{2g}$. See e.g. [FM12, 10.3] for more details.

Definition 3.13. *A sequence of Fuchsian representations ρ_n is normalized if the following occurs. We fix a set $\gamma_1, \dots, \gamma_{2g}$ of generators of $\pi_1 S$, and three distinct points a, b, c of $\partial_\infty \mathbb{H}^2$. Then we ask a to be the attractive fixed point of $\rho_n(\gamma_{2g})$, b to be the repulsive fixed point of $\rho_n(\gamma_{2g})$, and c to be the attractive fixed point of $\rho_n(\gamma_{2g-1})$ for all n .*

These are the Fricke coordinates of the Teichmüller space. We will need the following classical result.

Lemma 3.14. *Let $(\rho_n)_n$ be a sequence of normalized Fuchsian representations. There exist homeomorphisms $\tilde{\varphi}_n : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ that satisfy, for any $\gamma \in \pi_1 S$,*

$$(14) \quad \tilde{\varphi}_n \circ \rho(\gamma) = \rho_n(\gamma) \circ \tilde{\varphi}_n$$

and $(\tilde{\varphi}_n)_n$ converge to the identity map, uniformly on compact sets.

Proof. We know that $(\mathbb{H}^2 / \rho_n(\pi_1 S))_n$ converges in Teichmüller space. By a theorem of Teichmüller, there are $K(n)$ -quasiconformal homeomorphisms from $\mathbb{H}^2 / \rho_n(\pi_1 S)$ to $\mathbb{H}^2 / \rho(\pi_1 S)$ with $K(n) \rightarrow 1$ [IT92]. Their lifts satisfy (14). Moreover, due to the normalization we chose for the groups $\rho_n(\pi_1 S)$, the lifts fix three distinct points on the boundary at infinity of \mathbb{H}^2 . Under this normalization condition, up to extract a subsequence, the sequence of homeomorphisms converge uniformly [Ahl06, Theorem 1 p. 32], [IT92]. \square

Definition 3.15. *A Fuchsian H-convex function is a pair (u, ρ) , where u is a H-convex function and ρ is a Fuchsian representation of $\pi_1 S$ into $O_0(2, 1)$, such that for any $\gamma \in \pi_1 S$,*

$$(15) \quad u \circ \rho(\gamma) = u .$$

We say that a sequence of Fuchsian H-convex functions $(u_n, \rho_n)_n$ converges to a pair (u, ρ) , if u is a H-convex function, ρ a Fuchsian representation, $(u_n)_n$ converges to u and $(\rho_n)_n$ converges to ρ .

Note that if (u, ρ) is a Fuchsian H-convex function, then $\rho(\pi_1 S)$ acts by isometries on d_u . Moreover, if d_u is a distance, then the quotient gives a distance \bar{d}_u on the compact surface $\mathbb{H}^2 / \rho(\pi_1 S)$. As the latter is compact, we obtain the following.

Fact 3.16. *If (u, ρ) is a Fuchsian H-convex function and d_u a distance, then d_u is a complete distance on \mathbb{H}^2 .*

Lemma 3.17. *Let $(u_n, \rho_n)_n$ be a sequence of Fuchsian H-convex functions that converges to a pair (u, ρ) . Then (u, ρ) is a Fuchsian H-convex function.*

Proof. Let $y \in \mathbb{H}^2$ and $\gamma \in \pi_1 S$. Then, for every $\epsilon > 0$

$$(16) \quad |u(\rho(\gamma)y) - u(y)| \leq |u(\rho(\gamma)y) - u_n(\rho(\gamma)y)|$$

$$(17) \quad + |u_n(\rho(\gamma)y) - u_n(\rho_n(\gamma)y)|$$

$$(18) \quad + |u_n(\rho_n(\gamma)y) - u_n(y)|$$

$$(19) \quad + |u_n(y) - u(y)| < \epsilon.$$

In fact, for n large enough, $d_{\mathbb{H}^2}(\rho(\gamma)y, \rho_n(\gamma)y) \rightarrow 0$ as $n \rightarrow \infty$, and as the u_n are equi-Lipschitz on a sufficiently large compact set (Lemma 3.4), the absolute value at line (17) is smaller than $\epsilon/4$ for n large enough. Moreover the absolute value at line (18) is zero for every n by the $\rho_n(\pi_1 S)$ -invariance of u_n , and the absolute value at lines (16) and (19) are smaller than $\epsilon/4$ for n large enough by the uniform convergence of the u_n . Since $\epsilon > 0$ is arbitrary, this completes the proof. \square

Corollary 3.18. *Let (u_n, ρ_n) be Fuchsian H -convex functions such that*

- $(u_n, \rho_n)_n$ converges to a pair (u, ρ) ;
- there exist $\alpha, \beta > 0$ with $\alpha < u_n < \beta$;
- d_{u_n} are distances with Lipschitz shortest paths;
- d_{u_n} converge to d_u , uniformly on compact sets.

There is a subsequence such that, on any compact set of \mathbb{H}^2 , $d_{u_n}(\tilde{\varphi}_n(\cdot), \tilde{\varphi}_n(\cdot))$ uniformly converge to d_u , where $\tilde{\varphi}$ is given by Lemma 3.14.

Proof. By Lemma 3.14 and Corollary 3.11, $x \mapsto d_{u_n}(\tilde{\varphi}_n(x), x)$ uniformly converges to 0 (on the given compact set). By the triangle inequality,

$$d_{u_n}(\tilde{\varphi}_n(x), \tilde{\varphi}_n(y)) - d_u(x, y) \leq d_{u_n}(\tilde{\varphi}_n(x), x) + d_{u_n}(\tilde{\varphi}_n(y), y) + d_{u_n}(x, y) - d_u(x, y)$$

and the right-hand side is uniformly less than any $\epsilon > 0$ for n sufficiently large by the preceding argument and the assumption. On the other hand,

$$d_u(x, y) - d_{u_n}(\tilde{\varphi}_n(x), \tilde{\varphi}_n(y)) = d_u(x, y) - d_{u_n}(x, y) + d_{u_n}(x, y) - d_{u_n}(\tilde{\varphi}_n(x), \tilde{\varphi}_n(y)).$$

We know that $d_u(x, y) - d_{u_n}(x, y)$ is uniformly less than any $\epsilon > 0$ for n sufficiently large, and the term $d_{u_n}(x, y) - d_{u_n}(\tilde{\varphi}_n(x), \tilde{\varphi}_n(y))$ is less than

$$d_{u_n}(x, \tilde{\varphi}_n(x)) + d_{u_n}(y, \tilde{\varphi}_n(y)) + d_{u_n}(\tilde{\varphi}_n(x), \tilde{\varphi}_n(y)) - d_{u_n}(\tilde{\varphi}_n(x), \tilde{\varphi}_n(y))$$

that is also uniformly less than any $\epsilon > 0$ for n sufficiently large. \square

Proposition 3.19. *Under the assumptions of Corollary 3.18, up to extract a subsequence, $(\mathbb{H}^2/\rho_n(\pi_1 S), \bar{d}_{u_n})$ uniformly converge to $(\mathbb{H}^2/\rho(\pi_1 S), \bar{d}_u)$.*

Proof. Every sequence of representation can be normalized by composing on the left by hyperbolic isometries. Also, applying an isometry to a surface, we do not change its induced distance. Hence, in the statement of Proposition 3.19, we may assume that the sequence of representations is normalized.

The maps $\tilde{\varphi}_n$ from Lemma 3.14 induces homeomorphisms

$$\varphi_n : \mathbb{H}^2/\rho(\pi_1 S) \rightarrow \mathbb{H}^2/\rho_n(\pi_1 S).$$

Let C be a compact set of \mathbb{H}^2 such that, for any $p, q \in \mathbb{H}^2/\rho(\pi_1 S)$, there are lifts x, y of p, q respectively, such that

$$(20) \quad \bar{d}_u(p, q) = d_u(x, y).$$

By definition of $\tilde{\varphi}_n$, $\tilde{\varphi}_n(x)$ and $\tilde{\varphi}_n(y)$ are lifts of $\varphi_n(p)$ and $\varphi_n(q)$. In particular, $d_{u_n}(\tilde{\varphi}_n(x), \tilde{\varphi}_n(y)) \geq \bar{d}_{u_n}(\varphi_n(p), \varphi_n(q))$, and

$$(21) \quad \bar{d}_{u_n}(\varphi_n(p), \varphi_n(q)) - \bar{d}_u(p, q) \leq d_{u_n}(\tilde{\varphi}_n(x), \tilde{\varphi}_n(y)) - d_u(x, y).$$

Now let us look at $\bar{d}_u(p, q) - \bar{d}_{u_n}(\varphi_n(p), \varphi_n(q))$. For any $\gamma \in \pi_1 S$,

$$d_u(x, \rho(\gamma)y) \geq d_u(x, y)$$

and by Corollary 3.18, for any $\epsilon > 0$, if n is sufficiently large, uniformly on C ,

$$d_{u_n}(\tilde{\varphi}_n(x), \rho_n(\gamma)\tilde{\varphi}_n(y)) + \epsilon \geq d_{u_n}(\tilde{\varphi}_n(x), \tilde{\varphi}_n(y))$$

and as $\bar{d}_{u_n}(\varphi_n(p), \varphi_n(q))$ is the minimum over $\pi_1 S$ of the $d_{u_n}(\tilde{\varphi}_n(x), \rho_n(\gamma)\tilde{\varphi}_n(y))$, we obtain

$$\bar{d}_{u_n}(\varphi_n(p), \varphi_n(q)) + \epsilon \geq d_{u_n}(\tilde{\varphi}_n(x), \tilde{\varphi}_n(y)) .$$

This last equation and (20) give

$$(22) \quad \bar{d}_u(p, q) - \bar{d}_{u_n}(\varphi_n(p), \varphi_n(q)) \leq d_u(x, y) - d_{u_n}(\tilde{\varphi}_n(x), \tilde{\varphi}_n(y)) + \epsilon .$$

From (21), (22), and Corollary 3.18 applied for the compact set C , $|\bar{d}_{u_n}(\varphi_n(p), \varphi_n(q)) - \bar{d}_u(p, q)|$ is uniformly less than any positive number for n sufficiently large. \square

3.5. The polyhedral case.

Definition 3.20. A Fuchsian H -convex function (u, ρ) is polyhedral if the graph G_u is the boundary of the convex hull of the orbit for $\rho(\pi_1 S)$ of a finite number of points in $I^+(0)$.

Lemma 3.21. Let (u, ρ) be a polyhedral Fuchsian H -convex function. Then d_u is a complete distance, with Lipschitz shortest paths. Moreover, on the set of Lipschitz paths, $\mathcal{L}_u = L_{d_u}$.

Proof. It can be easily showed that G_u is a locally finite gluing of compact convex Euclidean polygons [Fil13]. In particular, d_u is a distance. Moreover, the quotient of G_u by $\rho(\pi_1 S)$ is compact, hence d_u is a complete metric. By Hopf–Rinow theorem, there is a shortest path between each pair of points. Clearly, a shortest path on G_u is a broken line, in particular, it is Lipschitz.

Let us consider a Lipschitz path on G_u . It is the union of a finite number of Lipschitz curves, each contained in a face of G_u . Those ones are compact convex polygons in Euclidean planes, where the Euclidean structure is the restriction of the Minkowski metric to the plane containing the face. It is well-known that in the Euclidean plane, the two ways of measuring length of Lipschitz curves coincide (see e.g. [Bur15] for a clear review of this fact and generalizations). \square

Putting together Lemma 3.3, Proposition 3.12 and Proposition 3.19, one obtains the following.

Proposition 3.22. Let $(u_n, \rho_n)_n$ be a sequence of polyhedral Fuchsian H -convex functions. If

- there are $\alpha, \beta > 0$ with $\alpha < u_n < \beta$,
- $(\rho_n)_n$ converge to a Fuchsian representation ρ ,

then, up to extract subsequences, there is a Fuchsian H -convex function (u, ρ) such that $(\mathbb{H}^2/\rho_n(\pi_1 S), \bar{d}_{u_n})_n$ uniformly converge to $(\mathbb{H}^2/\rho(\pi_1 S), \bar{d}_u)$.

4. COMPACTNESS RESULTS

The aim of this section is to give, in some sense, the converse to Proposition 3.22. More precisely, we want to prove the following.

Proposition 4.1. *Let (S, d) be a metric of non-positive curvature, and let (u_n, ρ_n) be polyhedral Fuchsian H -convex functions such that*

- *the representations ρ_n are normalized in the sense of Definition 3.13.*
- *the sequence of compact surfaces $(\mathbb{H}^2/\rho_n(\pi_1 S), \tilde{d}_{u_n})$ uniformly converges to (S, d) ;*

then, up to extract a subsequence,

- *there exist $\alpha, \beta > 0$ such that*

$$\alpha < u_n < \beta$$

- *the sequence ρ_n converge to a Fuchsian representation ρ .*

Under the hypothesis of Proposition 4.1, there are homomorphisms

$$\psi_n : \mathbb{H}^2/\rho_n(\pi_1 S) \rightarrow S$$

such that, if we denote by d_n the push-forward by ψ_n of $(\mathbb{H}^2/\rho_n(\pi_1 S), d_{u_n})$, then on S , the sequence $(d_n)_n$ uniformly converge to d . By equivariance, if \tilde{d}_n is the lift of d_n to the universal cover of S , and $\tilde{\psi}$ is a lift of ψ , for any $x, y \in \mathbb{H}^2$,

$$(23) \quad d_{u_n}(x, \rho_n(\gamma)y) = \tilde{d}_n(\tilde{\psi}(x), \gamma \cdot \tilde{\psi}(y)) .$$

4.1. Lower bound. In this Paragraph we will show that there is a uniform lower bound for the functions u_n , $n \in \mathbb{N}$. First, with (23), Corollary 2.14 translates as follows.

Corollary 4.2. *There exists $G > 0$ such that, for every element $\gamma \in \pi_1 S \setminus \{0\}$, for any n and for any $x \in \mathbb{H}^2$:*

$$(24) \quad d_{u_n}(x, \rho_n(\gamma)x) \geq G .$$

We also need the following lemma for the hyperbolic metrics.

Lemma 4.3. *There exists $R > 0$ (depending only on the genus of S) such that, for any $n \in \mathbb{N}$, for any $x \in \mathbb{H}^2$, there exists $\gamma_n \in \pi_1 S \setminus \{0\}$, such that*

$$d_{\mathbb{H}^2}(x, \rho_n(\gamma_n)x) < R .$$

Proof. Suppose the converse, i.e., for R arbitrary large there is however a point x and an index n such that for any $\gamma \in \pi_1 S \setminus \{0\}$ the inequality

$$(25) \quad d_{\mathbb{H}^2}(x, \rho_n(\gamma)x) \geq R$$

holds. Let us consider the Dirichlet polygon D for $\rho_n(\pi_1 S)$ centred at x . Let r be the radius of the largest disc centred at x contained in D . Then there is γ_x such that $r = d_{\mathbb{H}^2}(x, \rho_n(\gamma_x)x)/2$. By (25) we have that $r \geq R/2$. Therefore, the area of the disc would be arbitrarily large, that is impossible, as it is less than the area of D , and this one depends only on the genus of S . \square

Proposition 4.4. *Under the hypothesis of Proposition 4.1, there exists $\alpha > 0$ such that for any n and for any $x \in \mathbb{H}^2$,*

$$u_n(x) > \alpha .$$

Proof. Let us suppose the converse, i.e. that for an arbitrary $\epsilon > 0$, there exists n and $x_n \in \mathbb{H}^2$ with

$$(26) \quad u_n(x_n) < \epsilon .$$

Let the element $\gamma_n \in \pi_1 S$ from Lemma 4.3 such that

$$(27) \quad d_{\mathbb{H}^2}(x_n, \rho_n(\gamma_n)x_n) \leq R .$$

Let σ be the intersection of the graph of u_n with the Minkowski 2-plane which passes through the points x_n , $\rho_n(\gamma_n)x_n$, and the origin of the coordinate system in $\mathbb{R}^{2,1}$; also, let $L(\sigma)$ be its length. In particular,

$$(28) \quad d_{u_n}(x_n, \rho_n(\gamma_n)x_n) \leq L(\sigma) .$$

By construction, σ is a piecewise linear spacelike curve in a Minkowski 2-plane. Applying several times the inverse triangle inequality for Minkowski 2-plane (see (4)) and using (15), we get

$$(29) \quad L(\sigma) \leq \|u_n(x_n)\rho_n(\gamma_n)x_n - u_n(x_n)x_n\|_- .$$

Looking at Figure 1, we write down:

$$\|u_n(x_n)\rho_n(\gamma_n)x_n - u_n(x_n)x_n\|_- = \sqrt{2}u_n(x_n)(\cosh(d_{\mathbb{H}^2}(x_n, \rho_n(\gamma_n)x_n))) - 1)^{1/2} .$$

so with (28), (29), (24), (26) and (27),

$$G \leq \sqrt{2}\epsilon(\cosh(R) - 1)^{1/2}$$

that is impossible as ϵ is arbitrarily small. \square

4.2. Convergence of groups. Here we prove the main step in the proof of Theorem 1.3. We adapt an argument that was developed by J.-M. Schlenker in [Sch] for convex surfaces in de Sitter space.

Proposition 4.5. *Under the hypothesis of Proposition 4.1, up to extract a subsequence, the sequence $(\rho_n)_n$ converges to a Fuchsian representation ρ .*

Let us denote by P_n the graph of u_n , and by f_n the restriction to P_n of the squared distance from the origin in Minkowski space, i.e. for each $y \in P_n$ we define

$$(30) \quad f_n(y) = -\langle y, y \rangle_- .$$

The function f_n is positive and invariant under the action of $\rho_n(\pi_1 S)$, which acts cocompactly on P_n , hence it attains its extremal values.

Fact 4.6. *Let c be an arc-length parametrized shortest path on P_n .*

- (1) $(f_n \circ c)'$ has a positive jump at its singular points.
- (2) At non-singular points, $(f_n \circ c)'' = -2$ and $(f_n \circ c)^{(n)} = 0$ for $n \geq 3$.
- (3) $f_n \circ c$ is regular at local maxima.

Proof. (1) The first assertion follows because the singular point of $f_n \circ c$ are when c crosses an edge, and the result follows by convexity of P_n (note that clearly the intersection of a shortest path with an edge reduces to a single point).

- (2) If the point $c(0)$ is not singular, then locally c is a spacelike segment that can be written as $c(0) + tv$, $t \in (-\epsilon, \epsilon)$, and v a unit spacelike segment. So

$$f_n(c(t)) = -t^2 - 2\langle c(0), v \rangle_- t - \langle c(0), c(0) \rangle_- .$$

- (3) If f_n has a local maximum at t_M , then there is a small neighbourhood (a, b) of t_M such that the function f_n is regular on $(a, t_M) \cup (t_M, b)$, $f_n(c(t))' \geq 0$ for all $t \in (a, t_M)$, and $f_n(c(t))' \leq 0$ for all $t \in (t_M, b)$. Thus, once we assume that t_M is a singular point of the application f_n , then, by the first fact, $(f_n \circ c)'$ has a positive jump at t_M , and so there are points on (t_M, b) where $(f_n \circ c)'$ is strictly greater than 0 which leads to a contradiction. \square

Let $y_M(n)$ be a point on P_n where f_n attains its maximum. Let $x_M(n)$ be the image of $y_M(n)$ by the radial projection from $I^+(0)$ onto \mathbb{H}^2 . Among all the

maxima of f_n , $y_M(n)$ is chosen as follows. As (d_n) uniformly converge to d , by Lemma 2.5, there is a metric d_{sup} majorizing the d_n . We choose $y_M(n)$ such that all the $\tilde{\psi}(x_M(n))$ belong to the same Dirichlet fundamental region for \tilde{d}_{sup} .

Let $\gamma \in \pi_1 S$. Let $c : [0, l_\gamma(n)] \rightarrow P_n$ be an arc-length parametrized shortest path of P_n between $y_M(n)$ and $\rho_n(\gamma)y_M(n)$.

Lemma 4.7. *With the notations above,*

$$(31) \quad \int_0^{l_\gamma(n)} |\langle c, c' \rangle_-| \leq \frac{l_\gamma(n)^2}{2}.$$

Proof. To simplify the notation, let us denote $f_n \circ c$ by f in this proof. Let T be a point where f attains a local maximum. By Fact 4.6, $(f(T))' = 0$, hence the Taylor polynomial of the function f at T is

$$P_T^f(t) = f(T) - (t - T)^2.$$

As the function f can have positive jumps, $f(t) \geq P_T^f(t)$ for all t . Let the points t_- and t_+ be the closest points to T , $t_- < T < t_+$, where f attains local minima, in particular $f(t_-) < f(T)$ and $f(t_+) < f(T)$. So we have that

$$f(t_-) \geq P_T^f(t_-) = f(T) - (t_- - T)^2 \quad \text{and} \quad f(t_+) \geq P_T^f(t_+) = f(T) - (t_+ - T)^2,$$

or, in other words,

$$(32) \quad 0 < f(T) - f(t_-) \leq (T - t_-)^2 \quad \text{and} \quad 0 < f(T) - f(t_+) \leq (t_+ - T)^2.$$

By definition, f has global (and also local) maxima at $t = 0$ and $t = l_\gamma(n)$. Let us now introduce the decomposition $0 = T_0 < t_1 < T_1 < \dots < t_N < T_N = l_\gamma(n)$ of the segment $[0, l_\gamma(n)]$, where f attains local minima at t_i , $i = 1, \dots, N$, and local maxima at T_i , $i = 0, \dots, N$. Hence, at the points of differentiability of f , we have that $f'(t) \geq 0$ for all intervals $[t_i, T_i]$, and that $f'(t) \leq 0$ for all intervals $[T_i, t_{i+1}]$. Note also that $f'(t) = -2\langle c(t), c'(t) \rangle_-$ at all points where f is differentiable. Therefore, by (32), for all intervals $[t_i, T_i]$ and $[T_i, t_{i+1}]$ we get

$$(33) \quad \int_{t_i}^{T_i} |\langle c(t), c'(t) \rangle_-| dt = \int_{t_i}^{T_i} \frac{f'(t)}{2} dt = \frac{f(T_i) - f(t_i)}{2} \leq \frac{(T_i - t_i)^2}{2}$$

and

$$(34) \quad \int_{T_i}^{t_{i+1}} |\langle c(t), c'(t) \rangle_-| dt = \int_{T_i}^{t_{i+1}} \frac{-f'(t)}{2} dt = \frac{f(T_i) - f(t_{i+1})}{2} \leq \frac{(t_{i+1} - T_i)^2}{2}.$$

Hence

$$\int_0^{l_\gamma(n)} |\langle c, c' \rangle_-| = \sum_{i=1}^N \left[\int_{T_{i-1}}^{t_i} |\langle c, c' \rangle_-| + \int_{t_i}^{T_i} |\langle c, c' \rangle_-| \right] \leq \frac{1}{2} \sum_{i=1}^N [(t_i - T_{i-1})^2 + (T_i - t_i)^2]$$

and as for non-negative real numbers, $\sum_{j=1}^m (a_j)^2 \leq (\sum_{j=1}^m a_j)^2$, we arrive at

$$\int_0^{l_\gamma(n)} |\langle c, c' \rangle_-| \leq \frac{1}{2} \left[\sum_{i=1}^N [(t_i - T_{i-1}) + (T_i - t_i)] \right]^2 = \frac{[T_N - T_0]^2}{2} = \frac{l_\gamma(n)^2}{2}.$$

□

Corollary 4.8. *We have, for any n ,*

$$d_{\mathbb{H}^2}(x_M(n), \rho_n(\gamma)x_M(n)) \leq \frac{B_\gamma}{\alpha} + \frac{B_\gamma^2}{2\alpha^2},$$

where α is the uniform lower bound of the functions u_n given by Proposition 4.4, and B_γ is a constant depending only on γ .

Proof. By Lemma 3.9,

$$d_{\mathbb{H}^2}(x_M(n), \rho_n(\gamma)x_M(n)) \leq \frac{l_\gamma(n)}{\alpha} + \frac{1}{\alpha^2} \int_0^{l_\gamma(n)} |\langle c, c' \rangle_-| .$$

With (31), we obtain

$$d_{\mathbb{H}^2}(x_M(n), \rho_n(\gamma)x_M(n)) \leq \frac{l_\gamma(n)}{\alpha} + \frac{l_\gamma(n)^2}{2\alpha^2} .$$

Now, $l_\gamma(n) = d_{u_n}(y_M(n), \rho_n(\gamma)y_M(n))$ so from (23),

$$l_\gamma(n) = \tilde{d}_n(\tilde{\psi}(x_M(n)), \gamma.\tilde{\psi}(x_M(n))) \leq \tilde{d}_{\sup}(\tilde{\psi}(x_M(n)), \gamma.\tilde{\psi}(x_M(n))) .$$

As by definition, $\tilde{\psi}(x_M(n))$ are all belonging to a same compact set of \tilde{S} , the existence of B_γ follows \square

For $\gamma \in \pi_1 S$, let us denote by $L_{\rho_n}(\gamma)$ the length of the geodesic representative of γ in $\mathbb{H}^2/\rho_n(\pi_1 S)$. In particular, $L_{\rho_n}(\gamma) = \min_{x \in \mathbb{H}^2} d_{\mathbb{H}^2}(x, \rho_n(\gamma)x)$.

Corollary 4.9. *Let $\gamma \in \pi_1 S$. Then, there exists $B_\gamma > 0$ such that for any n ,*

$$(35) \quad L_{\rho_n}(\gamma) \leq \frac{B_\gamma}{\alpha} + \frac{B_\gamma^2}{2\alpha^2} .$$

We have proved that for every $\gamma \in \pi_1 S$, $L_{\rho_n}(\gamma)$ is bounded from above by a constant which does not depend on n . We can now use the following result.

Proposition 4.10 (Proposition 7.11 in [FcLP12]). *There exist $\gamma(i) \in \pi_1 S$, $i = 1, \dots, 9g-9$, such that the map from Teichmüller space of S to \mathbb{R}^{9g-9} that gives the length of the geodesic representatives of $\gamma(i)$ is injective and proper.*

Corollary 4.9 and Proposition 4.10 imply that the sequence of elements of Teichmüller space defined by $(\rho_n(\pi_1 S))_n$ is lying in a compact set, so up to extract a subsequence, the sequence $(\rho_n(\pi_1 S))_n$ converges in Teichmüller space. By the normalization we defined in Definition 3.13 for the groups $\rho_n(\pi_1 S)$, Proposition 4.5 follows.

4.3. Upper bound.

Proposition 4.11. *Under the hypothesis of Proposition 4.1, there exists $\beta > 0$ such that for any n and for any $x \in \mathbb{H}^2$,*

$$u_n(x) < \beta .$$

Proof. As the sequence of representation converges (Proposition 4.5), there exists a compact set $C \subset \mathbb{H}^2$ which contains a fundamental domain for $\rho_n(\pi_1 S)$ for all n . For each n , let x_n be a point of C which realizes the minimum of u_n . Suppose that $(u_n(x_n))_n$ is not bounded from above. In particular, up to extract a subsequence, u_n are uniformly bounded from below by a positive constant α , and moreover one can take α arbitrarily large in Corollary 4.9. But then, for any $\gamma \in \pi_1 S$, $L_{\rho_n}(\gamma)$ is arbitrarily small, that is impossible on a compact hyperbolic surface. For example, one can use the fact that only finitely many closed geodesics have length less than a given constant [Bus10, Theorem 1.6.11]. Hence there exists b such that for any n , $u_n(x_n) < b$.

From Proposition 4.4, $\alpha \leq u_n(x_n)$ independently of n . Let $y \in C$. As C is compact, there exists M such that, for any n ,

$$-\langle x_n, y \rangle_- = \cosh d_{\mathbb{H}^2}(x_n, y) \leq M .$$

As the surfaces defined by u_n are spacelike, by Lemma 4.12 below,

$$0 < \langle u_n(x_n)x_n - u_n(y)y, u_n(x_n)x_n - u_n(y)y \rangle_-.$$

Developing the right-hand side and using the bounds introduced above, we arrive at the condition

$$0 < -u_n(y)^2 + Mb u_n(y) - \alpha^2,$$

where the constants are independent of the choice of n and y in C . So the u_n are uniformly bounded from above. \square

Lemma 4.12. *For any x, y on a spacelike convex surface, $x \neq y$, the segment between x and y is spacelike.*

Proof. Assume the contrary, i.e. the segment $[x, y]$ is lightlike or timelike. Consider the intersection of the surface with the plane passing through the origin and x, y . It is a convex curve, which has a line parallel to $[x, y]$ as support line, hence there is a support plane of the surface containing this line, and this plane cannot be spacelike if this line is not spacelike, that is a contradiction. \square

5. PROOF OF THEOREM 1.3

Let us now consider the statement of Theorem 1.3. So let d be a metric with non-positive curvature on the compact surface S . By Corollary 2.21, there exists a sequence $(d_n)_{n \in \mathbb{N}}$ of polyhedral metrics with non-positive curvature on S that converges uniformly to d . By Theorem 1.5, for each $n \in \mathbb{N}$ there is a Fuchsian convex isometric immersion (ϕ_n, ρ_n) , such that $\phi_n(\tilde{S})$ is a convex polyhedral surface. Up to compose by global Minkowski isometries, we consider that the sequence of representations is normalized in the sense of Definition 3.13.

Then, if $\phi_n(\tilde{S})$ is the graph of the polyhedral Fuchsian H-convex function (u_n, ρ_n) , Proposition 4.1 applies: there is a subsequence of ρ_n converging to a Fuchsian representation ρ , and $\alpha, \beta > 0$ such that $\alpha < u_n < \beta$.

So Proposition 3.22 applies: the induced distance on the quotient of d_u by $\rho(\pi_1 S)$, say (S, m) , is the uniform limit of the (S, d_n) . The limit for uniform convergence (actually for the weaker Gromov-Hausdorff convergence) is unique (up to isometries) [BBI01], hence (S, m) is isometric to (S, d) . Theorem 1.3 is proved.

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